$1/f$ noise and one-dimensional Brownian motion in a singular potential

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A very simple model about the one-dimensional Brownian motion of a single particle in a singular potential field is proposed. In addition, the noise term is also weighted by a singular function, and this is the key to obtain $1/f$ noise for the present model. The power spectrum of the Brownian motion is investigated. The numerical calculation shows that the power spectrum has the form of $1/f$, which is different from ordinary Brownian motion.

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I. INTRODUCTION

$1/f$ noise has been observed in a wide variety of systems ranging from the current flowing through resistors and the light from quasars to the traffic current [1–3]. The widespread occurrence of these phenomena suggests that some underlying mechanism might exist. However, a satisfactory explanation has not been found yet and $1/f$ noise is still an open question. As we know, the noise spectrum of a "purely random" process is one associated with an autocorrelation function of the form $C(t,\tau)=\exp(-t/\tau)$, τ being the "relaxation time" of the process. Such a random process has a Lorentzian spectrum [4,5]. $1/f$ noise can be obtained by the superposition of the elementary processes with Lorentzian spectra and proper distribution of relaxation times (e.g., the $1/\tau$ distribution) in some complex systems. Recently, Bak, Tang, and Wiesenfeld [6] proposed a concept of self-organized criticality (SOC). They found that some complex systems can self-organize themselves spontaneously into a critical state with no intrinsic time nor length scale. Since temporal scale invariance immediately implies power spectra that behave like $1/f^\alpha$ for small f, SOC seems a promising mechanism for generating $1/f$ noise. Moreover, recent study shows that the fluctuations of certain physical quantities in some models [7–9] which display SOC indeed behave as $1/f$. These examples raise hopes for SOC as an underlying mechanism for generating $1/f$ noise in some systems.

However, in some material systems [1], alternative mechanisms have been proposed to explain $1/f$ noise. A common feature of these material systems is the presence of some form of disorder. Because of the disorder in these material systems, there exist some shallow or deep Coulombic centers [10], and these centers must affect the motion of charge carriers. It is believed that for many materials the underlying mechanism of $1/f$ noise involves the trapping or scattering of the charge carriers in the localized states [1,11,12]. Another phenomenon which exhibits $1/f$ noise [2] is related to the gravitational potential. Taking these two kinds of phenomena into account, we propose a model for Brownian motion of a single particle in a central potential. Although many models which associate $1/f$ noise with Brownian motion have been proposed, the models for Brownian motion of a single particle in a singular potential field have not been found. For simplicity, in this paper we consider only the one-dimensional case. In Sec. IV a generalized form of this model is also given.

II. THE MODEL FOR BROWNIAN MOTION IN A CENTRAL POTENTIAL

Our model is described by the following equation:

$$\dot{x}=-\frac{c\,\mathrm{sgn}(x)}{x^2}+\frac{\Gamma(t)}{x} \quad \text{for } x\neq 0\ , \tag{1}$$

where c is a positive constant and $\Gamma(t)$ is a Gaussian white noise with the properties

$$\langle\Gamma(t)\rangle=0$$

and

$$\langle\Gamma(t)\Gamma(t')\rangle=2D\delta(t-t')\ ,$$

where D is a constant. Without loss of generality, we may take $c=1$. Please note that the noise term is also weighted by a singular function, and this is the key to obtaining the $1/f$ noise for our present model. This term can be understood as the motion of a Brownian particle on the background of medium particles whose density distribution has the $1/|x|$ form because of the existence of the central potential. In a sense, the present model can be compared to the model of Ref. [8] [see Eq. (5) in Ref.

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[8]] where the deterministic term and the noise are weighted by functions which decay from the left boundary ($x_{\parallel}=0$) with some characteristic lengths.

It is known that the position $x(t)$ of a random walk has a $1/f^2$ power spectrum, because the velocity $\dot{x}$ has a white noise. Also, when the motion is described by the ordinary Langevin equation [e.g., $\dot{x}=f(x)+\Gamma(t)$; here $f(x)$ is an ordinary continuous function], the power spectrum of $x(t)$ cannot have the form of $1/f$. In some cases, it is a Lorentzian spectrum or a quasi-Lorentzian spectrum. However, the power spectrum of the position $x(t)$ in the present model with a singular potential and a noise weighted by a singular function is different. Obviously, when the second term on the right-hand side of Eq. (1) does not have the weighting factor and is only the noise $\Gamma(t)$, the probability density distribution of the particle will have the form $\exp(1/|x|)$. That is to say, because of the action of the singular potential, the particle will be attracted to the origin. In addition, we cannot obtain $1/f$ noise in this case. However, when the noise term is weighted by a singular function, the case is different. In order to know the general motion of the particle, we can write down the corresponding Fokker-Planck equation and look for the stationary solution. The Fokker-Planck equation for $x>0$ is

$$\frac{\partial}{\partial x}W(x,t)=-\left[\frac{\partial}{\partial x}D^{(1)}(x)-\frac{\partial^2}{\partial x^2}D^{(2)}(x)\right]W(x,t)\ , \tag{2}$$

with

$$D^{(1)}(x)=-\frac{1}{x^2}-\frac{D}{x^3},\quad D^{(2)}(x)=\frac{D}{x^2}\ .$$

Here $W(x,t)$ is the probability density distribution, and $D^{(1)}(x)$ and $D^{(2)}(x)$ are the drift and diffusion coefficients, respectively. Obviously, the above equation has a stationary solution, which has the form

$$W(x)\sim x\exp(-x/D)\quad \text{for } x>0\ . \tag{3a}$$

Similarly, we can also get the solution for $x<0$:

$$W(x)\sim -x\exp(x/D)\quad \text{for } x<0\ . \tag{3b}$$

From Eqs. (3a) and (3b) we can see that the probability is zero at the origin and it has the largest value at positions $x=\pm D$. Because of the probability that the Brownian particle at the origin is zero, in the numerical calculation we shall therefore deal with Eq. (1) only in the first approximation.

III. NUMERICAL CALCULATION AND RESULTS

The position $x(t+\Delta t)$ at the time $t+\Delta t$ can be obtained by integrating Eq. (1) from t to $t+\Delta t$. Integrating Eq. (1) reads as follows:

$$x^3(t+\Delta t)=x^3(t)-3\,\mathrm{sgn}[x(t)]\Delta t+3\int_t^{t+\Delta t}x(t')\Gamma(t')dt' \quad \text{if } |x^3(t)|>3\Delta t\ . \tag{4a}$$

If $|x^3(t)|\le 3\Delta t$, we let

$$x^3(t+\Delta t)=3\int_t^{t+\Delta t}x(t')\Gamma(t')dt'\ . \tag{4b}$$

When Δt is very small, we may consider only the first-order approximation

$$x^3(t+\Delta t)=\{x^3(t)-3\,\mathrm{sgn}[x(t)]\Delta t\}\Theta[\tfrac{1}{3}|x^3(t)|-\Delta t]+3x(t)\int_t^{t+\Delta t}\Gamma(t')dt'\ , \tag{5}$$

where

$$\Theta(x)=\begin{cases}0, & x<0\\ 1, & x\ge 0\ .\end{cases}$$

Following the algorithms for white noise used in Ref. [13], we use $\sqrt{2D\Delta t}\,\psi$ as a substitute for the last term $\int_t^{t+\Delta t}\Gamma(t')dt'$ in the above equation, where ψ is a random variable with properties

$$\langle\psi\rangle=0,\quad \langle\psi^2\rangle=1\ .$$

Then the above equation can be expressed as

$$x(t+\Delta t)=(\{x^3(t)-3\,\mathrm{sgn}[x(t)]\Delta t\}\Theta[\tfrac{1}{3}|x^3(t)|-\Delta t]+3x(t)\sqrt{(2D\Delta t)}\psi)^{1/3}\ . \tag{6}$$

As shown in Sec. II, the Brownian particle will reach the stationary state represented by Eqs. (3a) and (3b) at last, though its initial position $x(0)$ may be different. Actually, in the numerical calculation, after a long time transient, the power spectrum $S(f)$ of the position $x(t)$ is irrelevant to the initial position $x(0)$ of the particle. In our calculation, the power spectrum $S(f)$ of the position $x(t)$ is obtained by squaring the fast Fourier transformation of the position $x(t)$ recorded. Obviously, the upper cutoff frequency is determined by the time interval of every two successive data of the time series of the position $x(t)$ recorded. It must be noted that this time interval must be much larger than the value of time step Δt used in the numerical calculations. In our calculations the time interval is 100 times as large as the time step Δt. Therefore, the reliable range between the lower and upper cutoff frequencies is determined by the number of data and the time interval of every two successive data in the time sequence of the position $x(t)$ record. So, in order to find the property of the power spectrum at both high frequency and low frequency, the time interval of every two successive data $x(t)$ recorded should be very small, while the number of data must be as large as possible. In addition, because the Brownian particle has the largest probability at the position and the origin $x=0$ is a singular point of Eq. (1), in the numerical calculation, a choice of a comparable large value of the constant D can make the results closer to the solution of Eq. (1).

Because of the singularity of the present model, the fluctuations of position $x(t)$ and the corresponding power spectrum are different from those in the models described by the ordinary Langevin equations. First, let us see the fluctuations of $x(t)$ in the case of $D=0.1$. In the numerical calculation, we record the positions $x(t)$ with a time interval of 0.0002. Figure 1(a) shows the fluctuations of $x(t)$. This figure contains only 2000 data of the time sequence of $x(t)$. In Fig. 1(b) we show a magnification of a section of Fig. 1(a) in order to exhibit the self-similar structure of the curve. From Fig. 1 we can see that the particle is near the position $D=0.1$ or -0.1 during most of the time, which is in agreement with the analytical result of Eq. (1). The corresponding power spectrum is shown in Fig. 2. This figure is obtained by averaging ten samples, and smoothed by averaging over 0.05 unit of $\ln f$. In order to exhibit the power spectrum of the position over a wide range of frequency, every time sequence of $x(t)$ contains a total of 262 144 data. From Fig. 2 we can see that the power spectrum behaves as $1/f$ over a very wide range of frequency. So, we obtain $1/f$ noise through a very simple mechanism.

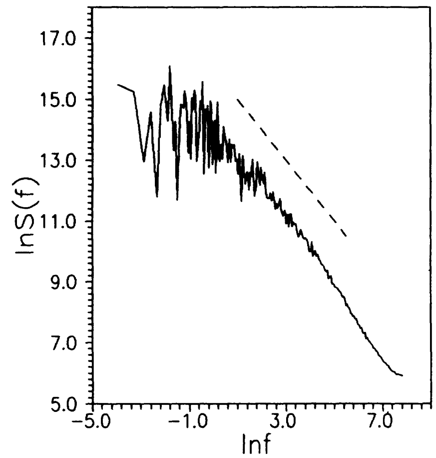

FIG. 2. The power spectrum of the position in the case $D=0.1$. The power spectrum behaves as $1/f$ over a wide range of frequency. The dashed line of slope -1 shows a $1/f$ dependence.

From Fig. 2 one can also see that there is a low-frequency cutoff (we denote this frequency as f_0) for $1/f$ behavior. It must be emphasized that the existence of f_0 in the present model is not due to the limits of computer time, but rather to the model itself. As mentioned above, because of the singularity of the deterministic force and the existence of the noise, the particle will vibrate near the origin after a long transient time. The fluctuation aroused by noise at a certain time would be completely forgotten after a sufficiently long time interval. Based on Eqs. (4a) and (4b), as an estimation of f_0, we can obtain the approximate value of f_0 through following expression:

$$x^3(t) \sim 3x(t)\sqrt{2D\Delta t_0}\psi_\Psi \sim 3\Delta t_0 , \tag{7}$$

that is to say, because of the action of the deterministic force, the particle has been back to the origin during Δt_0 before the next fluctuation begins. From the above expression we have

$$x^2(t) \sim 3\sqrt{2D\Delta t_0}\psi ,$$

$$x^4(t) \sim 9\times 2D\Delta t_0 \sim 6Dx^3(t) ,$$

i.e.,

$$x(t) \sim 6D .$$

So, we obtain

$$f_0 = \frac{1}{\Delta t_0} \sim \tfrac{1}{3}x^3(t) \sim 72D^3 . \tag{8}$$

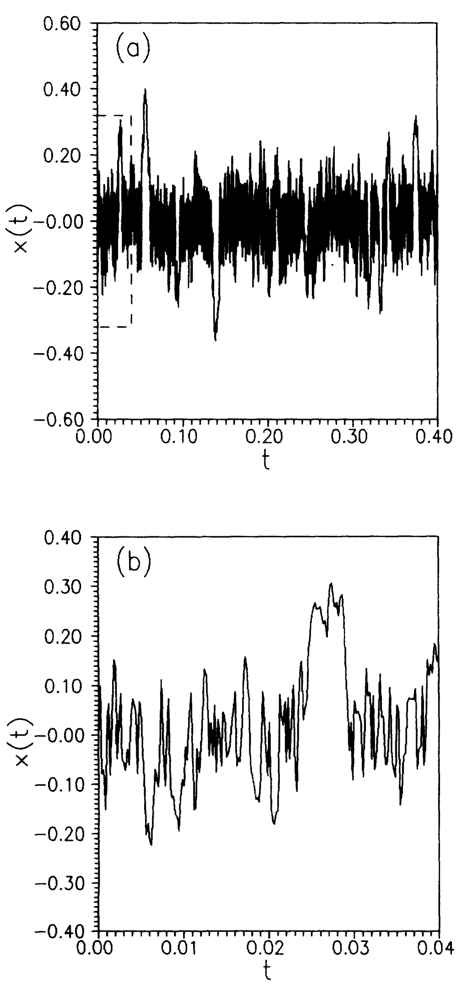

FIG. 1. (a) The fluctuations of the position $x(t)$ when $D=0.1$; here only the first 2000 data are shown. (b) Magnification of the rectangular region in (a).

IV. THE GENERALIZED MODEL AND RESULTS

From Sec. III we know that the singularity of the noise term is the key to obtaining the $1/f$ spectrum of noise in the present model. We now ask whether the concrete form of singularity also influences the results. Let us examine the properties of a more generalized version of the above model. The equation now reads

$$\dot{x} = -\frac{c\,\mathrm{sgn}(x)}{|x|^m} + \frac{\Gamma(t)}{x^n} \quad \text{for } x \neq 0 ; \tag{9}$$

here $\Gamma(t)$ has the same properties as in Sec. II and m and n are constants. Here we mainly discuss the power spectrum properties for the case of $m = n + 1$.

First, we discuss the D dependence of the fluctuations of $x(t)$ and the corresponding power spectrum. From Eq. (9) we obtain the following equation when $m = n + 1$:

$$\begin{aligned}[x(t+\Delta t)]^{n+2} &= [x(t)]^{n+2} - (n+2)\mathrm{sgn}[x(t)]\Delta t \\ &\quad + (n+2)x(t)\sqrt{(2D\Delta t)}\psi ,\end{aligned} \tag{10}$$

for Δt small enough. Now introduce a scale transformation, letting $x = \lambda\xi$, $t = \lambda^{\kappa}\tau$, and $D = \lambda^d D'$, and we have

$$\begin{aligned}\lambda^{n+2}\xi^{n+2}(\tau+\Delta\tau) &= \lambda^{n+2}\xi^{n+2}(\tau) - (n+2)\lambda^{\kappa}\Delta\tau\,\mathrm{sgn}[\xi(\tau)] \\ &\quad + (n+2)\lambda\xi(\tau)\sqrt{2\lambda^d D'\lambda^{\kappa}\Delta\tau}\psi\end{aligned}$$

or

$$\begin{aligned}\xi^{n+2}(\tau+\Delta\tau) &= \xi^{n+2}(\tau) - (n+2)\Delta\tau\,\mathrm{sgn}[\xi(\tau)]\lambda^{\kappa-n-2} \\ &\quad + (n+2)\xi(\tau)\sqrt{2D'\Delta\tau}\psi\lambda^{d/2+\kappa/2-n-1} .\end{aligned} \tag{11}$$

In order to keep the form of the equation unchanged, we may chose

$$\kappa = n + 2 \quad \text{and} \quad d = n .$$

So Eq. (4) is invariant under the following scale transformation:

$$x \to \lambda x, \quad t \to \lambda^{n+2} t, \quad D \to \lambda^n D . \tag{12}$$

This result means that the magnitude of the fluctuations of the position $x(t)$ changes as the value of D changes, but the shapes of the power spectra for different values of D are the same; only the frequency region of the power spectra becomes different. For instance, when $m = n + 1 = 2$ (the model of Sec. II), the magnitude of the fluctuations of $x(t)$ in the case of $D = 0.5$ is five times as large as one of the case $D = 0.1$ ($\lambda = 5$), and the frequency region of the power spectrum has a ln125 unit of left movement along the $\ln f$ axis in Fig. 2, but the shape of the power spectrum $\ln S(f)$ does not change.

Second, let us examine the properties of the power spectra in the case of $m = n + 1 > 1$. As in Sec. II, we can obtain the power spectra for different values of m. The power spectra are shown in Fig. 3 where we only give the results for $m = 1.0$, 1.2, 1.4, 1.8, 2.0, 4.0, 8.0, 16.0, 32.0, 64.0, and 128.0, and all curves have been smoothed by averaging over 0.1 interval of $\ln f$. From this figure one sees that when $m \geq 4.0$, the power spectra for different values of m almost overlap one another, and all exhibit $1/f^{\alpha}$ behavior with $\alpha \approx 1.28$ over a wide frequency range. So, in this case, the power spectra are insensitive to the degree of the singular potential. On the other hand, as shown in Fig. 3, when $m < 2$, the value α changes from 0 to 1.0. So the degree of the singular potential now has an essential influence upon the power spectrum.

Third, when $m = n$, Eq. (9) reduces to the ordinary Langevin equation. As an example, we calculated the power spectrum for $m = n = 2.0$ and find that it has the form $1/f^{\alpha}$ with $\alpha \approx 1.7$. That is to say, in this case, the power spectrum is almost that of ordinary Brownian motion. When $m < n$, the singular order of the noise term is higher than that of the deterministic term near the position $x = 0$. Hence, the motion of the particle near the origin is mainly controlled by the noise. So the motion of the particle is almost random, and the power spectrum of the position of the particle has the form $1/f^2$. Figure 4 shows the power spectrum in the case of

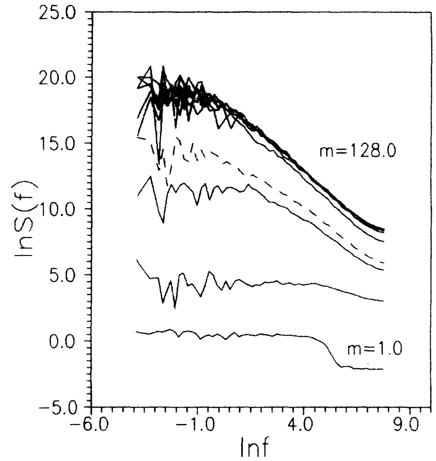

FIG. 3. The power spectra of the position for different values of m in the case of $m = n + 1$. Only the cases when m is equal to 1.0, 1.2, 1.4, 1.8, 2.0, 4.0, 8.0, 16.0, 32.0, 64.0, and 128.0, respectively, are given, and the dashed curve is given for the case of $m = 2.0$.

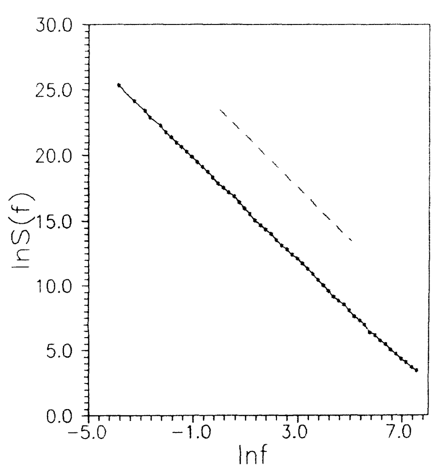

FIG. 4. The power spectrum of the position has the form $1/f^2$ for the case of $n = m + 1 = 2$. The dashed line shows a $1/f^2$ power spectrum for comparison.

$m+1=n=2$. From this figure, one sees that the power spectrum is indeed the same as that of Brownian motion.

V. DISCUSSION OF OUR CONCLUSION

In Eqs. (1) and (9), we can see that the present model contains two factors: one is the deterministic force, and the other is stochastic force (noise). Moreover, they are both singular in the origin. The deterministic force term makes the Brownian particle move toward the origin, but the action of the stochastic force term (noise) is to violate this tendency to the origin. The result of competition between these two kinds of force is that the Brownian particle fluctuates around the origin. Meanwhile, due to the singular weight of the noise, the particle may be moved away from the origin and stay a long time at one side of the origin before it moves to another side, and this is the reason for the existence of long-range correlations.

As shown in Sec. IV, when $m=n+1>2$, the shapes of the power spectra of the position are not very sensitive to the values of m. This demonstrates that the key to obtaining $1/f$ noise is the singularly weighted noise term and a proper deterministic term corresponding to noise. Although the practical difference between the values m and n may be different in the numerical simulations, the value m should be larger than the value n.

In brief, we proposed a simple model for a one-dimensional Brownian motion of the single particle in a singular potential, and discussed several simple examples. Numerical calculation shows that the power spectrum of the position for this kind of model may have the $1/f$ form over a large frequency range. Because this model contains only two terms and is very simple, it contributes to understanding of the mechanism of $1/f$ noise.

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